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From the special 2 + 1 Toda lattice to the Kadomtsev–Petviashvili equation

Cewen Cao†, Xianguo Geng† and Yongtang Wu‡

† Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, People's Republic of China
‡ Department of Computer Science, Hong Kong Baptist University, 224 Waterloo Road, Kowloon, Hong Kong, People's Republic of China

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Abstract. The nonlinearization of the eigenvalue problems associated with the Toda hierarchy and the coupled Korteweg–de Vries (KdV) hierarchy leads to an integrable symplectic map S and an integrable Hamiltonian system (H_0), respectively. It is proved that S and (H_0) have the same integrals $\{H_k\}$. The quasi-periodic solution of the (2 + 1)-dimensional Kadomtsev–Petviashvili equation is split into three Hamiltonian systems (H_0), (H_1), (H_2), while that of the special $(2+1)$-dimensional Toda equation is separated into (H_0), (H_1) plus the discrete flow generated by the symplectic map S. A clear evolution picture of various flows is given through the 'window' of Abel–Jacobi coordinates. The explicit theta-function solutions are obtained by resorting to this separation technique.

1. Introduction

In [1], the explicit quasi-periodic solutions of some (1 + 1)- as well as (2 + 1)-dimensional integrable models, such as the coupled nonlinear Schrödinger equation and the Kadomtsev–Petviashvili (KP) equation, are obtained in three steps:

(a) decomposition;
(b) straightening out of the flow;
(c) inversion.

The meaning of (a) is a nonlinear separation of variables, which reduces higher-dimensional integrable models into lower ones, and is realized by the so-called nonlinearization technique. Step (b) makes it possible to integrate the models simply and directly. In step (c) we write the explicit solutions in the original variables. Both (b) and (c) are completed by the algebro-geometric approach.

The aim of the present paper is to extend the method to discrete integrable models, with a special emphasis on the more difficult (2 + 1)-dimensional ones.

The decomposition of integrable models, or the nonlinear separation of variables, as the basis of all of the analysis, stems from the Lax representation of soliton equations. Integrable models, no matter whether they are continuous or discrete, 1 + 1 or 2 + 1 dimensional, are usually compatible conditions of certain overdetermined linear equations, which are called the Lax pair in the soliton literature. It is the nonlinearization of the Lax pair that provides an effective way to split the integral models into lower-dimensional ones, and finally into Hamiltonian flows or discrete symplectic flows in the phase space $\{\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q\}$ [1–4].

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In the 1 + 1 continuous case, every equation in a certain soliton hierarchy

$$\frac{du}{d\tau_k} = Y_k(u) \tag{1.1}$$

is usually expressed in the form of a zero-curvature equation:

$$\frac{dU}{d\tau_k} = V_{k,x} - [U, V_k] \tag{1.2}$$

which is the compatible condition of two overdetermined systems of linear equations (Lax pair)

$$\begin{aligned} \frac{d}{dx}\begin{pmatrix} p \\ q \end{pmatrix} &= U(u,\lambda)\begin{pmatrix} p \\ q \end{pmatrix} \\ \frac{d}{d\tau_k}\begin{pmatrix} p \\ q \end{pmatrix} &= V_k(u,\lambda)\begin{pmatrix} p \\ q \end{pmatrix}. \end{aligned} \tag{1.3}$$

It is interesting that there exists a relation between the potential u and the 'eigenfunction' p, q:

$$u = f_c(p, q) \tag{1.4}$$

which nonlinearizes (1.3) into two compatible Hamiltonian systems (after simply substituting (1.4) into (1.3)):

$$\begin{aligned} \frac{d}{dx}\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} -\partial H_0/\partial q \\ \partial H_0/\partial p \end{pmatrix} \\ \frac{d}{d\tau_k}\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} -\partial H_k/\partial q \\ \partial H_k/\partial p \end{pmatrix}. \end{aligned} \tag{1.5}$$

The system (H_0) is completely integrable in the Liouville sense [5], and $H_1, H_2, \ldots$ are exactly its integrals, involutive with each other (see [1]).

The procedure from (1.3) to (1.5) via (1.4) is called nonlinearization of the Lax pair and has the following three features:

(a) the linear equation (1.3) becomes nonlinear (1.5);
(b) the overdetermined equation (1.3) becomes compatible (1.5);
(c) the soliton equation (1.1), as a compatible condition of (1.3), becomes naturally satisfied by (1.4), so long as p, q is a compatible solution of (1.5).

Thus the 1 + 1 soliton equation (1.1) is decomposed (conditionally) into two 0 + 1 integrable models (1.5). In short, (Y_k) is split into (H_0) and (H_k).

This procedure is valid for almost all 1 + 1 soliton hierarchies known so far. Here (1.4) plays an essential role. The original motivation comes from Moser's investigation of the relation between the KdV hierarchy and the classical Neumann system (harmonic oscillator constrained on a sphere), where (1.4) is derived from the sphere condition, the so-called McKean–Trubowitz identity concerning the eigenfunctions of Hill's equation [6, 7]. Another powerful motivation comes from the scattering expression of the reflectionless potential [8], or the Bargmann potential in the KdV–Schrödinger case, where (1.4) is exactly the scattering expression. In [9] these two kinds of constraints were first summarized in the convenient form

$$G_{-1} = \sum_{j=1}^{N} \gamma_j \frac{\delta\lambda_j}{\delta u} \qquad \text{(Neumann constraint)} \tag{1.6}$$

$$G_0 = \sum_{j=1}^{N} \gamma_j \frac{\delta\lambda_j}{\delta u} \qquad \text{(Bargmann constraint)} \tag{1.7}$$

which can be regarded as symmetry constraints since $G_{-1}, G_0, \delta\lambda_j/\delta u$ are all symmetries of the given soliton equation (1.1). In the examples such as the KdV and the AKNS (Ablowitz, Kanp, Newell and Segur) hierarchies, we use the scattering expressions of the reflectionless potential as the starting point, which suggests the inner relation between the inverse scattering transform method (IST) and the algebro-geometric approach (see [1, 10, 11]).

As for the 1 + 1 discrete case, the first equation in (1.3) is replaced by a discrete eigenvalue problem, which is nonlinearized into a symplectic map S, instead of the first system of (1.5). In this case the discrete soliton model (X_k) is decomposed into (S) and (H_k).

The 2 + 1 soliton equations, both continuous and discrete, are much more complicated. Nevertheless, they could be decomposed in a similar procedure from their Lax representation into 1 + 1 dimensional equations [12–14], and further into 0 + 1 dimensional equations. Unfortunately, up to now the list of known 2 + 1 integrable models has been fairly short.

In the present paper we are going to investigate the Toda lattice (and one of its (2 + 1)-dimensional counterparts), which is an important discrete model with a physical background [15, 23–25]. Mathematically, it is the first member in the isospectral hierarchy of the discrete Toda eigenvalue problem:

$$Lq_j \equiv (aE + E^- a + b)q_j = \alpha_j q_j. \tag{1.8}$$

There are infinitely many members $\{X_k\}$ in the hierarchy, which commute with each other. The nonlinearization of (1.8) under the Bargmann constraint gives an integrable symplectic map S [2, 16–18] in $\{\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q\}$ with N integrals $H_0, H_1, \ldots, H_{N-1}$, which are independent and in involution with each other (see section 2.4).

It is discovered that the nonlinearization of the eigenvalue problem [19]

$$\partial_x \begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-\alpha_j + u) & v \\ -1 & \frac{1}{2}(\alpha_j - u) \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix} \tag{1.9}$$

associated with the coupled KdV (cKdV) hierarchy $\{Y_k\}$ leads to the same integrals $H_0, H_1, \ldots, H_{N-1}$ (see section 8).

According to the nonlinearization technique, the symplectic map S and the Hamiltonian systems H_0, H_1 and H_2 play the role of 'bricks', from which two 2 + 1 integrable models, the special (2 + 1)-dimensional Toda equation and the well known Kadomtsev–Petviashvili equation, are (conditionally) built up. Specifically, we have

$$\begin{tikzcd}
H_0 \arrow[dr, dash] & & \\
 & X_0 \arrow[dr, dash] & \\
S \arrow[ur, dash] \arrow[dr, dash] & & \text{special } 2+1 \text{ Toda} \\
 & X_1 \arrow[ur, dash] & \\
H_1 \arrow[ur, dash] & &
\end{tikzcd} \tag{1.10}$$

$$\begin{tikzcd}
H_1 \arrow[dr, dash] & & \\
 & Y_1 \arrow[dr, dash] & \\
H_0 \arrow[ur, dash] \arrow[dr, dash] & & \text{KP} \\
 & Y_2 \arrow[ur, dash] & \\
H_2 \arrow[ur, dash] & &
\end{tikzcd} \tag{1.11}$$

(see sections 2, 4 and 7). Though quite different at first glance, these two 2 + 1 models are linked up by the common basis as shown in the diagrams.

The Abel–Jacobi coordinate ϕ straightens out the H_k-flow as well as the discrete S-flow on the Jacobi variety (see sections 5 and 6):

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau_k} = \Omega_k \qquad k = 0, 1, 2, \ldots$$

$$\phi(n+1) - \phi(n) = \Omega_S$$

where τ_k and n are the associated flow variables. It is very easy to integrate these equations. Thus through the 'window' of the Abel–Jacobi coordinate we have a clear evolution picture of various flows, which means that the decomposition of special quasi-periodic solutions of nonlinear integrable models could be essentially reduced to a linear superposition:

$$\left.\begin{array}{l}\text{discrete } S\text{-flow}\\ \text{stationary Toda}\end{array}\right\} \qquad \phi = \phi_0 + n\Omega_S$$

$$\left.\begin{array}{l}\text{Bargmann flow for cKdV}\\ \text{stationary cKdV}\end{array}\right\} \qquad \phi = \phi_0 + x\Omega_0$$

$$H_k\text{-flow:} \quad \phi = \phi_0 + \tau_k\Omega_k$$

$$\text{Toda flow} X_k\text{:} \quad \phi = \phi_0 + n\Omega_S + \tau_k\Omega_k$$

$$\text{special 2 + 1 Toda:} \quad \phi = \phi_0 + n\Omega_S + x\Omega_0 + y\Omega_1$$

$$\text{cKdV flow } Y_k\text{:} \quad \phi = \phi_0 + x\Omega_0 + \tau_k\Omega_k$$

$$\text{KP flow:} \quad \phi = \phi_0 + x\Omega_0 + y\Omega_1 + t\Omega_2.$$

The explicit solutions expressed by the theta function for these equations (mainly theorems 7.2 and 8.8 for 2 + 1 Toda and 2 + 1 KP, respectively) are obtained through the Abel–Jacobi–Riemann inversion. Some of the results coincide with those found in [1, 15, 20] (see sections 7 and 8). Section 8 is brief since the continuous model of KP is not our main concern here. For more detail see [1], where there is a quite similar treatment. It is interesting to point out that both the AKNS hierarchy (in [1]) and the coupled KdV hierarchy (in the present paper), though quite different, lead to the same KP equation.

Another application of the decomposition diagram is that it provides an effective way in numerical analysis and graphical representation of solutions of the integrable nonlinear models (see [2, 4]).

For a deeper understanding of the Toda lattices, Kodama's recent work is very interesting [23–25].

2. The Toda hierarchy

Let E be the shift operator: $Ef(n) = f(n+1)$, $E^-f(n) = f(n-1)$ and $\Delta = E - 1$, $\Delta^- = 1 - E^-$. Denote

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Toda lattice eigenvalue problem (1.8) is put in the form

$$E\begin{pmatrix} p_j \\ q_j \end{pmatrix} = U(a, b, \alpha_j)\begin{pmatrix} p_j \\ q_j \end{pmatrix} \tag{2.1}$$

$$U(a, b, \lambda) = \frac{1}{a}\begin{pmatrix} 0 & a^2 \\ -1 & \lambda - b \end{pmatrix} \tag{2.2}$$

by introducing $p_j = E^- a q_j$.

Lemma 2.1 (Fundamental identity [2]). *Let $\sigma(a, b, \lambda)$ be a linear map defined as*

$$V = \sigma(a, b, \lambda)[\gamma] = -\left\{\tfrac{1}{2}\Delta^- a\gamma^{(1)} + (b - \lambda)\gamma^{(2)}\right\}\sigma_1 - \left(2E^- a^2\gamma^{(2)}\right)\sigma_2 + 2\gamma^{(2)}\sigma_3. \tag{2.3}$$

Then the discrete commutative relation

$$(EV)U - UV = U_*\begin{pmatrix} a \\ b \end{pmatrix}\{-(K - \lambda J)\gamma\} \tag{2.4}$$

holds for any function $\gamma = (\gamma^{(1)}, \gamma^{(2)})^T$, where

$$K = \begin{pmatrix} \frac{1}{2}a(\Delta + \Delta^-)a & a\Delta b \\ b\Delta^- a & 2(a^2\Delta + \Delta^- a^2) \end{pmatrix} \qquad J = \begin{pmatrix} 0 & a\Delta \\ \Delta^- a & 0 \end{pmatrix} \tag{2.5}$$

$$U_*\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} \delta a \\ \delta b \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} U(a + \varepsilon\delta a, b + \varepsilon\delta b, \lambda) = \frac{1}{a^2}\begin{pmatrix} 0 & a^2\delta a \\ \delta a & (b - \lambda)\delta a - a\delta b \end{pmatrix}. \tag{2.6}$$

Corollary 2.2. $(K - \lambda J)\gamma = 0$ *implies* $\det\sigma[\gamma] = \text{constant}$, *independent of n.*

Proof. By (2.4), $V = \sigma[\gamma]$ satisfies $V_{n+1} = UV_nU^{-1}$. Thus $\det V_{n+1} = \det V_n$. □

The Lenard gradients $\{g_k\}$ are a universal polynomial of a, b:

$$g_{-2} = \begin{pmatrix} a_n^{-1} \\ 0 \end{pmatrix} \qquad g_{-1} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad g_0 = \frac{1}{2}\begin{pmatrix} 2a_n \\ b_n \end{pmatrix}$$
$$g_1 = \frac{1}{2}\begin{pmatrix} 2a_n(b_{n+1} + b_n) \\ a_n^2 + a_{n-1}^2 + b_n^2 \end{pmatrix} \qquad \text{etc} \tag{2.7}$$

satisfying the recursive formula

$$Kg_{-2} = Jg_{-2} = 0 \qquad Jg_{-1} = 0 \qquad Kg_{j-1} = Jg_j \tag{2.8}$$

which means that $(K - \lambda J)g_\lambda = 0$ for the generating function

$$g_\lambda = g_{-1} + \sum_{k=0}^{\infty} g_k\lambda^{-k-1}. \tag{2.9}$$

By corollary 2.2, $\det\sigma[g_\lambda] = \text{constant}$. Since the Lenard gradients are universal polynomials of a and b, this constant can be determined by considering in the class of a, b with rapidly decaying condition as $n \to \infty$. We have

$$\det\sigma[g_\lambda] = -\tfrac{1}{4}\lambda^2 \tag{2.10}$$

after taking into account the structure of (2.3). The discrete Toda equation is defined as

$$\frac{\mathrm{d}}{\mathrm{d}\tau_k}\begin{pmatrix} a \\ b \end{pmatrix} = X_k = Jg_k. \tag{2.11}$$

The first two members are ($\tau_0 = x, \tau_1 = y$)

$$\frac{d}{dx}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = X_0(n) = \begin{pmatrix} \frac{1}{2}a_n(b_{n+1} - b_n) \\ a_n^2 - a_{n-1}^2 \end{pmatrix} \tag{2.12}$$

$$\frac{d}{dy}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = X_1(n) = \begin{pmatrix} \frac{1}{2}a_n(a_{n+1}^2 - a_{n-1}^2 + b_n^2 - b_{n-1}^2) \\ a_n^2(b_{n+1} + b_n) - a_{n-1}^2(b_n + b_{n-1}) \end{pmatrix}. \tag{2.13}$$

Let

$$a_n = \exp\tfrac{1}{2}(Q_{n+1} - Q_n) \qquad b_n = \frac{d}{dx}Q_n. \tag{2.14}$$

Then (2.12) is transformed into the usual Toda equation

$$\frac{d^2 Q_n}{dx^2} = \exp(Q_{n+1} - Q_n) - \exp(Q_n - Q_{n-1}). \tag{2.15}$$

The compatible solution of (2.12) and (2.13) yields the solution to the special (2 + 1)-dimensional Toda equation:

$$\frac{\partial^2 Q_n}{\partial x \partial y} = \exp(Q_{n+1} - Q_n)\frac{\partial}{\partial x}(Q_{n+1} + Q_n) - \exp(Q_n - Q_{n-1})\frac{\partial}{\partial x}(Q_n + Q_{n-1}). \tag{2.16}$$

3. The integrable symplectic map

In the continuous case, the nonlinearization of the eigenvalue problem gives an integrable system, while in the discrete case it yields an integrable symplectic map.

Lemma 3.1.

$$\nabla\alpha_j = \begin{pmatrix} \delta\alpha_j/\delta a \\ \delta\alpha_j/\delta b \end{pmatrix} = \begin{pmatrix} q_j E q_j \\ \frac{1}{2}q_j^2 \end{pmatrix} \tag{3.1}$$

$$\sigma(a, b, \alpha_j)[\nabla\alpha_j] = \begin{pmatrix} p_j q_j & -p_j^2 \\ q_j^2 & -p_j q_j \end{pmatrix} \equiv \varepsilon_j \tag{3.2}$$

$$(E\varepsilon_j)U - U\varepsilon_j = 0 \tag{3.3}$$

$$(K - \alpha_j J)\nabla\alpha_j = 0. \tag{3.4}$$

Proof. Equation (3.1) is a well known fact. Equations (3.2) and (3.3) are results of direct calculations. According to (2.4), equation (3.4) is equivalent to (3.3) since the linear map U_* is one-to-one. □

Let $\lambda = \alpha_1, \ldots, \alpha_N$ be N distinct eigenvalues. Put the N copies of eigenvalue problems (2.1) in vector form

$$Ep = aq \qquad Eq = a^{-1}r \equiv \frac{1}{a}(Aq - p - bq) \tag{3.5}$$

where $p = (p_1, \ldots, p_N)^T$, $q = (q_1, \ldots, q_N)^T$, $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$. Consider the Bargmann constraint

$$g_0 = \sum_{j=1}^{N} \nabla\alpha_j \tag{3.6}$$

which is equivalent to

$$a = \langle q, Eq\rangle = \sqrt{\langle q, r\rangle} \qquad b = \langle q, q\rangle$$

or

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{\langle Aq, q\rangle - \langle p, q\rangle - \langle q, q\rangle^2} \\ \langle q, q\rangle \end{pmatrix} \equiv f_S(p, q). \tag{3.7}$$

The eigenvalue problem (3.5) is nonlinearized by f_S into the map

$$E\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} aq \\ a^{-1}r \end{pmatrix} = \begin{pmatrix} aq \\ a^{-1}(Aq - p - \langle q, q\rangle q) \end{pmatrix} \equiv S\begin{pmatrix} p \\ q \end{pmatrix} \tag{3.8}$$

with $a = \sqrt{\langle Aq, q\rangle - \langle p, q\rangle - \langle q, q\rangle^2}$.

Proposition 3.2. *S is a symplectic map in* $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$.

Proof. Let $\tilde{p} = aq, \tilde{q} = a^{-1}r$. A direct calculation gives $\mathrm{d}\tilde{p} \wedge \mathrm{d}\tilde{q} = \mathrm{d}p \wedge \mathrm{d}q$. □

In order to prove the integrability of S, we consider

$$G_\lambda = g_{-1} + \sum_{j=1}^{N} \frac{\nabla\alpha_j}{\lambda - \alpha_j} = \begin{pmatrix} Q_\lambda(q, Eq) \\ \frac{1}{2}\{1 + Q_\lambda(q, q)\} \end{pmatrix} \tag{3.9}$$

$$V_\lambda = \sigma(\lambda)[G_\lambda] = \begin{pmatrix} \frac{1}{2}\lambda & -\langle p, q\rangle \\ 1 & -\frac{1}{2}\lambda \end{pmatrix} + \begin{pmatrix} Q_\lambda(p, q) & -Q_\lambda(p, p) \\ Q_\lambda(q, q) & -Q_\lambda(p, q) \end{pmatrix} \tag{3.10}$$

$$F_\lambda = \det V_\lambda \tag{3.11}$$

where (3.10) is obtained by direct calculations resorting to (3.2) with

$$Q_\lambda(\xi, \eta) = \langle(\lambda - A)^{-1}\xi, \eta\rangle = \sum_{j=1}^{N} \frac{\xi_j\eta_j}{\lambda - \alpha_j}.$$

It is easy to prove that under the Bargmann constraint (3.6), $(K - \lambda J)G_\lambda = 0$. Hence by (2.4) the Lax matrix V_λ satisfies

$$(EV_\lambda)U - UV_\lambda = 0. \tag{3.12}$$

According to corollary 2.2, F_λ is invariant under the symplectic map S and yields the integrals $\{F_j\}$ as follows:

$$\begin{aligned} F_\lambda &= \left\{Q_\lambda(p, p) + \langle p, q\rangle\right\}\left\{Q_\lambda(q, q) + 1\right\} - \left\{Q_\lambda(p, q) + \tfrac{1}{2}\lambda\right\}^2 \\ &= -\tfrac{1}{4}\lambda^2 + Q_\lambda(p, p) + \langle p, q\rangle Q_\lambda(q, q) - Q_\lambda(Ap, q) + Q_\lambda(p, p)Q_\lambda(q, q) - Q_\lambda^2(p, q) \\ &= -\tfrac{1}{4}\lambda^2 + \sum_{k=0}^{\infty} \lambda^{-k-1}F_k \end{aligned} \tag{3.13}$$

where

$$F_0 = \langle p, p\rangle + \langle p, q\rangle\langle q, q\rangle - \langle Ap, q\rangle$$

$$\begin{aligned} F_k = {} & \langle A^k p, p\rangle + \langle p, q\rangle\langle A^k q, q\rangle - \langle A^{k+1} p, q\rangle \\ & + \sum_{i+j=k-1} \{\langle A^i p, p\rangle\langle A^j q.q\rangle - \langle A^i p, q\rangle\langle A^j p, q\rangle\}. \end{aligned}$$

In order to prove the involutivity of $\{F_k\}$, we introduce the generating function method, which is convenient in a series of later calculations. Denote the variable of F_λ-flow by t_λ. Then

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} -\partial F_\lambda/\partial q_k \\ \partial F_\lambda/\partial p_k \end{pmatrix} = W(\lambda, \alpha_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix}$$
$$W(\lambda, \mu) = \frac{2}{\lambda - \mu} V_\lambda - V_\lambda^{21}\sigma_1. \tag{3.14}$$

Proposition 3.3.

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda} V_\mu = [W(\lambda, \mu), V_\mu] \qquad \forall\lambda \quad \mu \in \mathbb{C} \tag{3.15}$$
$$(F_\mu, F_\lambda) = 0 \qquad \forall\lambda \quad \mu \in \mathbb{C} \tag{3.16}$$
$$(F_j, F_k) = 0 \qquad \forall j \quad k = 0, 1, 2, \ldots. \tag{3.17}$$

Proof. A direct calculation gives (3.15), which implies the invariance of $F_\mu = \det V_\mu$ along the t_λ-flow:

$$0 = \frac{\mathrm{d}F_\mu}{\mathrm{d}t_\lambda} = (F_\mu, F_\lambda).$$

The expansion of (3.16) according to the negative powers of λ, μ gives (3.17). □

4. Decomposition of the Toda equation

Consider another generating function H_λ defined as the squared root of the normalized F_λ:

$$(1 + 4H_\lambda)^2 = -\frac{4}{\lambda^2} F_\lambda = 1 - \sum_{k=0}^{\infty} \frac{4F_k}{\lambda^{k+3}}. \tag{4.1}$$

It is easy to find the recursive formula for the polynomials determined by H_λ:

$$H_\lambda = \sum_{k=0}^{\infty} \frac{H_k}{\lambda^{k+3}}$$
$$H_m = -\tfrac{1}{2} F_m \qquad m = 0, 1, 2 \tag{4.2}$$
$$H_{k+3} = -\tfrac{1}{2} F_{k+3} - 2 \sum_{\substack{i+j=k \\ i,j\geqslant 0}} H_i H_j \qquad k = 0, 1, 2, \ldots.$$

Exerting $J^{-1}K$ on the Bargmann constraint (3.6) k times gives

$$\sum_{j=1}^{N} \alpha_j^k \nabla\alpha_j = g_k + c_2 g_{k-2} + \cdots + c_{k+1} g_{-1} + c'_{k+2} g_{-2} \tag{4.3}$$

where there are two extra terms $cg_{-1} + c'g_{-2}$ each time, since the linear space $\ker J$ is two dimensional with the generators g_{-1} and g_{-2}.

Proposition 4.1. *f_S maps the solution of the discrete flow*

$$\begin{pmatrix} p(n) \\ q(n) \end{pmatrix} = S^n \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \tag{4.4}$$

into the solution $(a, b)^T = f_S(p, q)$ of the stationary Toda equation

$$X_N + c_{N1} X_{N-1} + \cdots + c_{NN} X_0 = 0. \tag{4.5}$$

Proof. The linear combination of (4.3) gives

$$0 = \sum_{j=1}^{N} \alpha(\alpha_j)\nabla\alpha_j = g_N + c_{N1}g_{N-1} + \cdots + c_{N,N+2}g_{-2} \tag{4.6}$$

where

$$\alpha(\lambda) = \prod_{j=1}^{N}(\lambda - \alpha_j) = \sum_{k=0}^{N} a_{N-k}\lambda^k. \tag{4.7}$$

Acting with J on (4.6) yields (4.5). □

Multiplied by λ^{-k-1} and summed with respect to k from 0 to ∞, equation (4.3) becomes

$$G_\lambda = c_\lambda g_\lambda + c'_\lambda g_{-2} \tag{4.8}$$

where

$$c_\lambda = 1 + \sum_{k=0}^{\infty} c_{k+2}\lambda^{-k-2} \qquad c'_\lambda = 1 + \sum_{k=0}^{\infty} c'_{k+3}\lambda^{-k-2}. \tag{4.9}$$

Hence

$$V_\lambda = c_\lambda \sigma[g_\lambda] \tag{4.10}$$

$$F_\lambda = c_\lambda^2 \det \sigma[g_\lambda] = -\tfrac{1}{4}\lambda^2 c_\lambda^2 \tag{4.11}$$

$$c_\lambda = 1 + 4H_\lambda \tag{4.12}$$

where (2.10) is used. By (4.2) we have

$$c_2 = 0 \qquad c_k = 4H_{k+3}. \tag{4.13}$$

Lemma 4.2. *Let* $(a, b)^T = f_S(p, q)$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} a \\ b \end{pmatrix} = -2JG_\lambda. \tag{4.14}$$

Proof. By (2.5) and (3.9), we obtain

$$JG_\lambda = \begin{pmatrix} \frac{1}{2}a\Delta Q_\lambda(q, q) \\ \Delta Q_\lambda(p, q) \end{pmatrix}.$$

A direct calculation shows that the two components of both sides of (4.14) are equal to

$$\frac{1}{a}\{[a^2 - (\lambda - b)^2]Q_\lambda(q, q) + 2(\lambda - b)Q_\lambda(p, q) - Q_\lambda(p, p) + [a^2 - \langle p, q\rangle + \lambda b - b^2]\}$$

$$2(b - \lambda)Q_\lambda(q, q) + 4Q_\lambda(p, q) + 2b.$$ □

Theorem 4.3.

(a)

$$\mathrm{d}f_S(I\nabla H_k) = X_k. \tag{4.15}$$

(b) Let $(p(n, \tau_k), q(n, \tau_k))^T$ *be the compatible solution of the* H_k*-flow (with the variable* τ_k*) and the discrete flow generated by the symplectic map* S *(with the variable* n*). Then* f_S *maps it into the solution of the* k*th Toda equation:*

$$\frac{\mathrm{d}}{\mathrm{d}\tau_k}\begin{pmatrix} a \\ b \end{pmatrix} = X_k \qquad k = 0, 1, 2, \ldots \tag{4.16}$$

where $(a, b)^T$ *is calculated by (3.7).*

Proof. Let τ_λ be the variable of the H_λ-flow. According to (4.1) and (4.12), it is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda} = \frac{-1}{2\lambda^2 c_\lambda}\frac{\mathrm{d}}{\mathrm{d}t_\lambda}. \tag{4.17}$$

By (4.14), we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\lambda^2} J g_\lambda.$$

Hence we have (4.16). □

5. Straightening out of the H_k-flow

Factorize F_λ, V_λ^{12} and V_λ^{21} as rational functions of λ:

$$F_\lambda = -V_\lambda^{12}V_\lambda^{21} - (V_\lambda^{11})^2 = \frac{-\beta(\lambda)}{4\alpha(\lambda)} = \frac{-R(\lambda)}{4\alpha^2(\lambda)} \tag{5.1}$$

$$V_\lambda^{12} = -Q_\lambda(p,p) - \langle p, q\rangle = -\langle p, q\rangle\frac{m(\lambda)}{\alpha(\lambda)} \tag{5.2}$$

$$V_\lambda^{21} = 1 + Q_\lambda(q,q) = \frac{n(\lambda)}{\alpha(\lambda)} \tag{5.3}$$

where

$$\alpha(\lambda) = \prod_{j=1}^{N}(\lambda - \alpha_j) \qquad \beta(\lambda) = \prod_{j=1}^{N+2}(\lambda - \beta_j)$$

$$R(\lambda) = \alpha(\lambda)\beta(\lambda) = \prod_{j=1}^{2N+2}(\lambda - \lambda_j)$$

$$m(\lambda) = \prod_{j=1}^{N}(\lambda - \mu_j) \qquad n(\lambda) = \prod_{j=1}^{N}(\lambda - \nu_j)$$

with $\lambda_j = \alpha_j$, $j = 1, \ldots, N$; $\lambda_{N+j} = \beta_j$, $j = 1, \ldots, N+2$. $\{\mu_j\}$ and $\{\nu_j\}$ are called elliptic coordinates. By comparing the coefficients of λ^{-k} in the expansions of (5.2) and (5.3) we have

$$\frac{\langle p, p\rangle}{\langle p, q\rangle} = \sum_{j=1}^{N}(\alpha_j - \mu_j) \tag{5.4}$$

$$\langle q, q\rangle = \sum_{j=1}^{N}(\alpha_j - \nu_j) \tag{5.5}$$

$$\langle Aq, q\rangle = \tfrac{1}{2}\sum_{j=1}^{N}\left(\alpha_j^2 - \nu_j^2\right) + \tfrac{1}{2}\left(\sum_{j=1}^{N}(\alpha_j - \nu_j)\right)^2. \tag{5.6}$$

In order to have the evolution of the elliptic coordinates along the t_λ-flow, we use the components of the Lax equation (3.14):

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V_\mu^{12} = -2W_{\lambda\mu}^{12}V_\mu^{11} + 2W_{\lambda\mu}^{11}V_\mu^{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V_\mu^{21} = 2W_{\lambda\mu}^{21}V_\mu^{11} - 2W_{\lambda\mu}^{11}V_\mu^{21}$$

and (5.1) with $\lambda = \mu_k, \nu_k$:

$$V_{\mu_k}^{11} = \frac{\sqrt{R(\mu_k)}}{2\alpha(\mu_k)} \qquad V_{\nu_k}^{11} = \frac{\sqrt{R(\nu_k)}}{2\alpha(\nu_k)}.$$

A direct calculation gives

$$\frac{1}{2\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}t_\lambda} = \frac{m(\lambda)}{\alpha(\lambda)(\lambda-\mu_k)m'(\mu_k)} \tag{5.7}$$

$$\frac{1}{2\sqrt{R(\nu_k)}}\frac{\mathrm{d}\nu_k}{\mathrm{d}t_\lambda} = \frac{-n(\lambda)}{\alpha(\lambda)(\lambda-\nu_k)n'(\nu_k)}. \tag{5.8}$$

Resorting to the interpolation formula, we have $(j = 1, 2, \ldots, N)$

$$\sum_{k=1}^{N} \frac{\mu_k^{N-j}}{2\sqrt{R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}t_\lambda} = \frac{\lambda^{N-j}}{\alpha(\lambda)} \tag{5.9}$$

$$\sum_{k=1}^{N} \frac{\nu_k^{N-j}}{2\sqrt{R(\nu_k)}}\frac{\mathrm{d}\nu_k}{\mathrm{d}t_\lambda} = \frac{-\lambda^{N-j}}{\alpha(\lambda)}. \tag{5.10}$$

Consider the algebraic curve Γ given by the affine equation, $\xi^2 - R(\lambda) = 0$, with genus $g = N$ and the usual holomorphic differentials

$$\tilde{\omega}_j = \frac{\lambda^{g-j}\,\mathrm{d}\lambda}{2\sqrt{R(\lambda)}} \qquad j = 1, \ldots, g. \tag{5.11}$$

Denote $P(\lambda) = (\lambda, \xi = \sqrt{R(\lambda)})$. For fixed point P_0 on Γ, introduce the quasi-Abel–Jacobi coordinates by

$$\tilde{\psi}_j = \sum_{k=1}^{g}\int_{P_0}^{P(\mu_k)} \tilde{\omega}_j \qquad \tilde{\phi}_j = \sum_{k=1}^{g}\int_{P_0}^{P(\nu_k)} \tilde{\omega}_j \qquad j = 1, \ldots, g. \tag{5.12}$$

Then (5.9) and (5.10) are rewritten as

$$\frac{\mathrm{d}\tilde{\psi}_j}{\mathrm{d}t_\lambda} = \frac{\lambda^{g-j}}{\alpha(\lambda)} \qquad \frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}t_\lambda} = \frac{-\lambda^{g-j}}{\alpha(\lambda)}. \tag{5.13}$$

Hence

$$\frac{\mathrm{d}\tilde{\psi}_j}{\mathrm{d}\tau_\lambda} = \frac{-\lambda^{g-j}}{2\lambda\sqrt{R(\lambda)}} \qquad \frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}\tau_\lambda} = \frac{\lambda^{g-j}}{2\lambda\sqrt{R(\lambda)}} \tag{5.14}$$

by (4.17), (4.12), (5.1) and (4.11).

Let $a_1, b_1, \ldots, a_g, b_g$ be the canonical basis of the homology group of cycles on Γ, and

$$C = (A_{jk})_{g\times g}^{-1} \qquad A_{jk} = \int_{a_k} \tilde{\omega}_j. \tag{5.15}$$

For the normalized holomorphic differential

$$\omega = C\tilde{\omega} \qquad \omega_s = \sum_{j=1}^{g} C_{sj}\tilde{\omega}_j \tag{5.16}$$

we have

$$\int_{a_k} \omega_j = \delta_{jk} \qquad \int_{b_k} \omega_j = B_{jk} \tag{5.17}$$

where the matrix $B = (B_{jk})$ is symmetric and has a positive-definite imaginary part and is used to construct the Riemann theta function of Γ [21]:

$$\theta(\zeta) = \sum_{z\in\mathbb{Z}^g} \exp \pi\sqrt{-1}(\langle Bz, z\rangle + 2\langle\zeta, z\rangle) \qquad \zeta \in \mathbb{C}^g.$$

The Abel map $\mathcal{A}$: $\mathrm{Div}(\Gamma) \to \mathcal{J}(\Gamma) = \mathbb{C}^g/\mathcal{T}$ is defined by

$$\mathcal{A}(P) = \int_{P_0}^{P} \omega \qquad \mathcal{A}\left(\sum n_k P_k\right) = \sum n_k \mathcal{A}(P_k) \tag{5.18}$$

where $\mathrm{Div}(\Gamma)$ is the divisor group, and the lattice $\mathcal{J}$ is spanned by the periodic vectors $\{\delta_j, B_j\}$, which are the column vectors of the unit matrix and B. Introduce the Abel–Jacobi coordinates

$$\begin{aligned} \psi &= \mathcal{A}\left(\sum_{k=1}^{g} P(\mu_k)\right) = C\tilde{\psi} \\ \phi &= \mathcal{A}\left(\sum_{k=1}^{g} P(\nu_k)\right) = C\tilde{\phi}. \end{aligned} \tag{5.19}$$

Through direct calculations we have the following assertions.

Lemma 5.1. *Let* $S_k = \lambda_1^k + \cdots + \lambda_{2g+2}^k$. *Then the coefficients in*

$$\frac{\lambda^{g+1}}{\sqrt{R(\lambda)}} = \sum_{k=0}^{\infty} \Lambda_k \lambda^{-k} \tag{5.20}$$

satisfy the recursive formula

$$\begin{aligned} &\Lambda_0 = 1 \qquad \Lambda_1 = \tfrac{1}{2}S_1 \\ &\Lambda_k = \frac{1}{2k}\left(S_k + \sum_{\substack{i+j=k \\ i,j\geqslant 1}} S_i\Lambda_j\right). \end{aligned} \tag{5.21}$$

Lemma 5.2. *Let* $C_1, \ldots, C_g$ *be the column vectors of* C. *Then the coefficient of the expansion*

$$\frac{\lambda^{g+1}}{2\sqrt{R(\lambda)}}(C_1\lambda^{-1} + \cdots + C_g\lambda^{-g}) = \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-1} \tag{5.22}$$

is written as

$$\Omega_k = \tfrac{1}{2}(\Lambda_k C_1 + \Lambda_{k-1}C_2 + \cdots + \Lambda_{k-g+1}C_g) \tag{5.23}$$

if we defined $\Lambda_{-s} = 0$, $s = 0, 1, 2, \ldots$. *Specifically,*

$$\Omega_0 = \tfrac{1}{2}C_1 \qquad \Omega_1 = \tfrac{1}{2}(\Lambda_1 C_1 + C_2) \qquad \Omega_2 = \tfrac{1}{2}(\Lambda_2 C_1 + \Lambda_1 C_2 + C_3).$$

Theorem 5.3. *The* H_k*-flow is straightened out by the Abel–Jacobi coordinates:*

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau_k} = -\Omega_k \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau_k} = \Omega_k. \tag{5.24}$$

Proof. By (5.19), (5.14) and (5.22) we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau_\lambda} = -\sum_{k=0}^{\infty} \Omega_k \lambda^{-k-3} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau_\lambda} = \sum_{k=0}^{\infty} \Omega_k \lambda^{-k-3} \tag{5.25}$$

which imply (5.24) since for arbitrary f we have

$$\frac{\mathrm{d}f}{\mathrm{d}\tau_\lambda} = (f, H_\lambda) = \sum_{k=0}^{\infty}(f, H_k)\lambda^{-k-3} = \sum_{k=0}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}\tau_k}\lambda^{-k-3}. \qquad \square$$

6. Straightening out of the discrete flow

In this section p and q are designated as scalars, not N-dimensional vectors. It would not cause any confusion since no p and q are contained in the final results of this section. Consider the Toda eigenvalue problem

$$\chi(n+1) = U_n\chi(n) \qquad U_n = \frac{1}{a_n}\begin{pmatrix} 0 & a_n^2 \\ -1 & \lambda - b_n \end{pmatrix} \tag{6.1}$$

where $\chi(n) = (p(n), q(n))^T$. The fundamental solution matrix

$$M_n = (\chi^{(1)}(n), \chi^{(2)}(n)) = \begin{pmatrix} p^{(1)}(n) & p^{(2)}(n) \\ q^{(1)}(n) & q^{(2)}(n) \end{pmatrix} \qquad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

can be expressed explicitly as

$$M_{n+1} = U_nU_{n-1}\ldots U_0. \tag{6.2}$$

By mathematical induction, it is easy to prove that

$$\begin{aligned}
M_1 &= \frac{1}{a_0}\begin{pmatrix} 0 & a_0^2 \\ -1 & \lambda - b_0 \end{pmatrix} \\
M_2 &= \frac{1}{a_0a_1}\begin{pmatrix} -a_1^2 & a_1^2(\lambda - b_0) \\ -(\lambda - b_0) & (\lambda - b_1)(\lambda - b_0) - a_0^2 \end{pmatrix} \\
p^{(1)}(n) &= \frac{-a_{n-1}}{a_0\ldots a_{n-2}}\left\{\lambda^{n-2} - \left(\sum_{j=1}^{n-2} b_j\right)\lambda^{n-3} + \cdots\right\} \\
p^{(2)}(n) &= \frac{a_{n-1}}{a_0\ldots a_{n-2}}\left\{\lambda^{n-1} - \left(\sum_{j=1}^{n-2} b_j\right)\lambda^{n-2} + \cdots\right\} \\
q^{(1)}(n) &= \frac{-1}{a_0\ldots a_{n-1}}\left\{\lambda^{n-1} - \left(\sum_{j=1}^{n-1} b_j\right)\lambda^{n-2} + \cdots\right\} \\
q^{(2)}(n) &= \frac{1}{a_0\ldots a_{n-1}}\left\{\lambda^{n} - \left(\sum_{j=1}^{n-1} b_j\right)\lambda^{n-1} + \cdots\right\}.
\end{aligned} \tag{6.3}$$

The Lax matrix V_λ of the symplectic map S defined by (3.10) satisfies the discrete Lax equation (3.12), which implies that the solution space of the linear equation $E\chi = U\chi$ is invariant under the action of V_λ. Let ρ be the eigenvalue of V_λ in the solution space, and χ be the eigenfunction, which is called the Baker function (after some normalization):

$$E\chi = U\chi \qquad V_\lambda\chi = \rho\chi. \tag{6.4}$$

Evidently, $\det|\rho - V_\lambda| = \rho^2 + F_\lambda = 0$. Thus there are two eigenvalues $\rho^\pm = \pm\rho$, whereby (5.1), (4.11), (4.12):

$$\rho = \frac{\lambda}{2}(1 + 4H_\lambda) = \frac{\sqrt{R(\lambda)}}{2\alpha(\lambda)} = \frac{\lambda}{2} + \mathrm{O}\left(\frac{1}{\lambda^2}\right). \tag{6.5}$$

Hence,

$$\begin{aligned}
V_\lambda^{11} + \rho &= \lambda + Q_\lambda(p,q) + 2\lambda H_\lambda = \lambda + \mathrm{O}\left(\frac{1}{\lambda}\right) \\
V_\lambda^{11} - \rho &= Q_\lambda(p,q) - 2\lambda H_\lambda = \frac{\langle p, q\rangle}{\lambda} + \mathrm{O}\left(\frac{1}{\lambda^2}\right).
\end{aligned} \tag{6.6}$$

An elementary discussion shows that the corresponding Baker functions can be taken as

$$\chi^{\pm}(n) = c^{\pm}\chi^{(1)}(n) + \chi^{(2)}(n)$$
$$c^{\pm} = \frac{V_\lambda^{11}(0) \pm \rho}{V_\lambda^{21}(0)}. \tag{6.7}$$

Lemma 6.1. *Let* $(EV_\lambda)U - UV_\lambda = 0$. *Then*

$$V_\lambda(n)M_n = M_nV_\lambda(0). \tag{6.8}$$

Proof.

$$\begin{aligned} V_\lambda(n) &= U_{n-1}V_\lambda(n-1)U_{n-1}^{-1} \\ &= U_{n-1}\ldots U_0V_\lambda(0)U_0^{-1}\ldots U_{n-1}^{-1} \\ &= M_nV_\lambda(0)M_n^{-1}. \end{aligned}$$
□

Proposition 6.2 (Formula of Dubrovin–Novikov type).

$$q^+(n,\lambda)q^-(n,\lambda) = \frac{V_\lambda^{21}(n)}{V_\lambda^{21}(0)} = \prod_{j=1}^{N} \frac{\lambda - \nu_j(n)}{\lambda - \nu_j(0)} \tag{6.9}$$

where q^+ *and* q^- *are the second components of (6.7).*

Proof. Through direct calculation with the use of (6.8), we obtain (6.9). □

$q^+(n,\lambda)$ and $q^-(n,\lambda)$ can be considered as values of $q(n, P)$ on the upper and lower sheets of Γ, respectively. The function $[\rho] = [R(\lambda)]^{1/2}/2\alpha(\lambda)$ has the values $\sqrt{R(\lambda)}/2\alpha(\lambda)$ and $-\sqrt{R(\lambda)}/2\alpha(\lambda)$ on the upper and lower sheet, respectively. With the coordinate $z = \lambda^{-1}$, $\hat{\xi} = \xi z^{N+1}$, we have the equation of Γ near infinity:

$$\hat{\xi}^2 - R_*(z) = 0 \qquad R_*(z) = z^{2N+2}R(z^{-1}) = (1-\lambda_1 z)\ldots(1-\lambda_{2N+2}z).$$

There are two infinities $\infty_s = (z = 0, \hat{\xi} = (-1)^s)$, $s = 1, 2$, which are located on the upper ($s = 2$) and lower ($s = 1$) sheet, respectively. By (6.3), (6.6) and (6.9), we have ($\lambda \to \infty$)

$$q^-(n,\lambda) = \frac{\lambda^n}{a_0\ldots a_{n-1}} + \mathrm{O}\big(\lambda^{n-1}\big)$$
$$q^+(n,\lambda) = a_0\ldots a_{n-1}\lambda^{-n} + \mathrm{O}\big(\lambda^{-n-1}\big).$$

Resorting to these asymptotic behaviours and (6.7), (6.9), through an elementary analysis we have:

Proposition 6.3. *The second component* $q(n, P)$ *of the Baker function has:*

(a) g *simple poles at* $\nu_1(0), \ldots, \nu_g(0)$ *and a zero of the* n*th order at* ∞_2;
(b) g *simple zeros at* $\nu_1(n), \ldots, \nu_g(n)$ *and a pole of the* n*th order at* ∞_1.

Consider the meromorphic differential on Γ

$$\omega_S(n) = \left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln q(n, P)\right\}\mathrm{d}\lambda \tag{6.10}$$

with the residue $-1, 1$ at $\nu_j(0), \nu_j(n)$, respectively, and the residue $-n, n$ at ∞_1, ∞_2, respectively. Decompose (6.10) as a linear combination

$$\omega_S(n) = \Omega + n\omega(\infty_2, \infty_1) + \sum_{j=1}^{g}\omega[\nu_j(n), \nu_j(0)] + \sum_{j=1}^{g}\gamma_j\omega_j \tag{6.11}$$

where ω_j is the normalized differential of the first kind given by (5.16), Ω is an Abel differential of the second kind and $\omega(P, Q)$ is the normal differential of the third kind with the residue $1, -1$ at P, Q, respectively, and the properties (see [15])

$$\int_{a_j}\omega(P, Q) = 0 \tag{6.12}$$

$$\int_{b_j}\omega(P, Q) = 2\pi\sqrt{-1}\int_Q^P\omega_j. \tag{6.13}$$

The integral of (6.10) along a_i gives $\gamma_i = 2\pi\sqrt{-1}\,n_i$, while the integral of (6.11) along b_i yields

$$\sum_{j=1}^{g}\int_{\nu_j(0)}^{\nu_j(n)}\omega = n\int_{\infty_2}^{\infty_1}\omega + \sum_{j=1}^{g}\left(n_jB_j + m_j\delta_j\right) \tag{6.14}$$

where n_j and m_j are certain integers. Thus we obtain

Theorem 6.4.

$$\Delta\phi = \phi(n+1) - \phi(n) = \int_{\infty_2}^{\infty_1}\omega \equiv \Omega_S \quad (\mathrm{mod}\,\mathcal{T}). \tag{6.15}$$

7. Algebro-geometric solution of the special 2 + 1 Toda equation

Lemma 7.1. *Near ∞_s, in the local coordinate $z = \lambda^{-1}$, we have*

$$\omega = (-1)^{s+1}\sum_{k=0}^{\infty}\Omega_k z^k\,\mathrm{d}z \tag{7.1}$$

$$\mathcal{A}(P(\lambda)) = -\eta_s + (-1)^{s+1}\sum_{k=0}^{\infty}\frac{1}{k+1}\Omega_k z^{k+1} \tag{7.2}$$

where

$$\eta_s = \int_{\infty_s}^{P_0}\omega. \tag{7.3}$$

Proof. We have

$$\omega = C\tilde{\omega} = \frac{\lambda^g\,\mathrm{d}\lambda}{2[R_(\lambda)]^{1/2}}(C_1\lambda^{-1} + \cdots + C_g\lambda^{-g}).$$

Near ∞_s, $[R(\lambda)]^{1/2}$ takes the value $(-1)^s\sqrt{R(\lambda)}$. By (5.22) and through direct calculations, we obtain the required results. $\square$

From (3.7) and (5.5) we have the formula for the potential b:

$$b = \langle q, q\rangle = A_1 - \sum_{j=1}^{g} \nu_j \tag{7.4}$$

where $A_1 = \sum \alpha_j$. In order to calculate $\sum \nu_j$, we use the Riemann theorem [21], which asserts that there exists a constant vector $\mathcal{K}$ such that $\theta(\mathcal{A}(P(\lambda)) - \phi - \mathcal{K})$ has exactly g zeros at $\lambda = \nu_1, \ldots, \nu_g$. Thus by a standard treatment [1, 21, 22], we have

$$\begin{aligned}\sum_{j=1}^{g} \nu_j &= I_1(\Gamma) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda\, \mathrm{d}\ln\theta(\mathcal{A}(P(\lambda)) - \phi - \mathcal{K}) \\ &= I_1(\Gamma) - \sum_{j=1}^{g} \Omega_0^j \frac{\partial}{\partial\zeta_j} \ln\frac{\theta(\phi + \mathcal{K} + \eta_2)}{\theta(\phi + \mathcal{K} + \eta_1)}\end{aligned} \tag{7.5}$$

where

$$I_k(\Gamma) = \sum_{j=1}^{g} \int_{a_j} \lambda^k \omega_j. \tag{7.6}$$

Theorem 7.2. *The special* $(2 + 1)$*-dimensional Toda equation (2.16) has the quasi-periodic solution*

$$\begin{aligned}Q_n(x, y) = \ln &\frac{\theta\{(n + 1)\Omega_S + x\Omega_0 + y\Omega_1 + D_1\}\theta\{n\Omega_S + y\Omega_1 + D_1\}}{\theta\{n\Omega_S + x\Omega_0 + y\Omega_1 + D_1\}\theta\{(n + 1)\Omega_S + y\Omega_1 + D_1\}} \\ &+\{A_1 - I_1(\Gamma)\}x + Q_n(0, y)\end{aligned} \tag{7.7}$$

where $D_1 = \phi_0 + \mathcal{K} + \eta_1$.

Proof. By the discussion with (2.16) and theorem 4.3, we have the decomposition diagram (1.3). Hence from theorem 5.3 and 6.4 we obtain the explicit solution written in the Abel–Jacobi coordinate: $\phi = n\Omega_S + x\Omega_0 + y\Omega_1 + \phi_0$, which is inverted by the above procedure into

$$b_n(x, y) = \partial_x \ln\frac{\theta\{(n + 1)\Omega_S + x\Omega_0 + y\Omega_1 + D_1\}}{\theta\{n\Omega_S + x\Omega_0 + y\Omega_1 + D_1\}} + A_1 - I_1(\Gamma)$$

where $\Omega_S = \eta_2 - \eta_1$ is used. The relation $b_n = \partial_x Q_n$ implies (7.7). $\square$

8. The coupled KdV hierarchy and the KP equation

The canonical equations of the Hamiltonian

$$H_0 = -\tfrac{1}{2}F_0 = -\tfrac{1}{2}\langle p, p\rangle - \tfrac{1}{2}\langle p, q\rangle\langle q, q\rangle + \tfrac{1}{2}\langle Ap, q\rangle \tag{8.1}$$

defined by (4.2) and (3.13) can be put in the form

$$\begin{aligned}&\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = \hat{U}\begin{pmatrix} p_j \\ q_j \end{pmatrix} \\ &\hat{U} = \tfrac{1}{2}(-\alpha_j + u)\sigma_1 + v\sigma_2 - \sigma_3\end{aligned} \tag{8.2}$$

with

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \langle q, q\rangle \\ \langle p, q\rangle \end{pmatrix} = f_c(p, q). \tag{8.3}$$

Equation (8.2) is exactly the eigenvalue problem which determine the cKdV soliton hierarchy [19]. We list the basic facts without proof as follows, which have quite a similar structure to those in the Toda case.

Proposition 8.1 (The fundamental identity). *Let* $(\partial = \partial/\partial_x)$

$$\hat{V} = \hat{\sigma}(u, v, \lambda)[\gamma] = \left\{\tfrac{1}{2}(-\partial - u + \lambda)\gamma^{(2)}\right\}\sigma_1 + \left\{\partial\gamma^{(1)} - v\gamma^{(2)}\right\}\sigma_2 + \gamma^{(2)}\sigma_3. \tag{8.4}$$

Then

$$\hat{V}_x - \left[\hat{U}, \hat{V}\right] = \hat{U}_*\left\{-(\hat{K} - \lambda\hat{J})\gamma\right\} \tag{8.5}$$

where

$$\hat{K} = \begin{pmatrix} 2\partial & \partial^2 + \partial u \\ -\partial^2 + u\partial & v\partial + \partial v \end{pmatrix} \qquad \hat{J} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{8.6}$$

The Lenard gradients are

$$\begin{aligned} &\hat{g}_{-2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{g}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \hat{g}_0 = \begin{pmatrix} v \\ u \end{pmatrix} \qquad \hat{g}_1 = \begin{pmatrix} -v_x + 2uv \\ u_x + u^2 + 2v \end{pmatrix} \\ &g_2 = \begin{pmatrix} v_{xx} - 3uv_x + 3u^2v + 3v^2 \\ u_{xx} + 3uu_x + u^3 + 6uv \end{pmatrix} \qquad \text{etc} \end{aligned} \tag{8.7}$$

with

$$\det\hat{\sigma}[\hat{g}_\lambda] = -\tfrac{1}{4}\lambda^2 \qquad \hat{g}_\lambda = \sum_{k=0}^{\infty}\hat{g}_{k-1}\lambda^{-k}. \tag{8.8}$$

The cKdV vector field is defined as $Y_j = Jg_j$ *with*

$$\begin{aligned} &Y_0 = \begin{pmatrix} u_x \\ v_x \end{pmatrix} \qquad Y_1 = \begin{pmatrix} u_{xx} + 2uu_x + 2v_x \\ -v_{xx} + 2uv_x + 2u_xv \end{pmatrix} \\ &Y_2 = \begin{pmatrix} u_{xxx} - 3uu_{xx} + 3u_x^2 + 3u^2u_x + 6u_xv + 6uv_x \\ v_{xxx} - 3uv_{xx} - 3u_xv_x + 3u^2v_x + 6uu_xv + 6vv_x \end{pmatrix}. \end{aligned} \tag{8.9}$$

Proposition 8.2. *Let* (u, v) *be the compatible solution of the* Y_1*- and* Y_2*-flow:*

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = Y_1 \qquad \begin{pmatrix} u \\ v \end{pmatrix}_t = Y_2. \tag{8.10}$$

Then $w = 2v$ *satisfies the KP equation*

$$w_{tx} = \tfrac{1}{4}\left(w_{xx} + 3w^2\right)_{xx} + \tfrac{3}{4}w_{yy}. \tag{8.11}$$

Proof. Equation (8.11) is based on the results of the calculations

$$v_{tx} = \left(v_{xx} + 3v^2 - 3uv_x + 3u^2v\right)_{xx}$$

$$\tfrac{3}{4}v_{yy} = \left(\tfrac{3}{4}v_{xx} + \tfrac{3}{2}v^2 - uv_x + 3u^2v\right)_{xx}. \qquad \square$$

Proposition 8.3.

$$\hat{\nabla}\alpha_j = \begin{pmatrix} \delta\alpha_j/\delta u \\ \delta\alpha_j/\delta v \end{pmatrix} = \begin{pmatrix} p_j q_j \\ q_j^2 \end{pmatrix} \tag{8.12}$$

$$\hat{\sigma}(u, v, \alpha_j)\hat{\nabla}\alpha_j = \begin{pmatrix} p_j q_j & -p_j^2 \\ q_j^2 & -p_j q_j \end{pmatrix} \equiv \varepsilon_j \tag{8.13}$$

$$\varepsilon_{j,x} - [\hat{U}(\alpha_j), \varepsilon_j] = 0 \tag{8.14}$$

$$(\hat{K} - \alpha_j \hat{J})\hat{\nabla}\alpha_j = 0. \tag{8.15}$$

The constraint (8.3) could be put in Bargmann form

$$\hat{g}_0 = \sum_{j=1}^{N} \hat{\nabla}\alpha_j. \tag{8.16}$$

Just as in (3.9) we construct

$$\hat{G}_\lambda = \hat{g}_{-1} + \sum_{j=1}^{N} \frac{\hat{\nabla}\alpha_j}{\lambda - \alpha_j} = \begin{pmatrix} Q_\lambda(p, q) \\ 1 + Q_\lambda(q, q) \end{pmatrix}. \tag{8.17}$$

Proposition 8.4. *The symplectic map S and the Bargmann system* $(H_0) = (8.2) + (8.16)$ *have the same Lax matrix* $V_\lambda = \hat{V}_\lambda$ *and the same conserved integrals* $\{F_j\}$ *or* $\{H_j\}$.

Proof. A direct calculation shows that

$$\hat{V}_\lambda = \hat{\sigma}_\lambda(\hat{G}_\lambda) = \tfrac{1}{2}(-u + \lambda)\sigma_1 - v\sigma_2 + \sigma_3 + \tfrac{1}{2}\sum_{j=1}^{N} q_j^2 \sigma_1 + \sum_{j=1}^{N} \frac{\varepsilon_j}{\lambda - \alpha_j}$$

which coincides with (3.10) as $u = \langle q, q \rangle$. □

Proposition 8.5. f_c *defined by (8.3) maps the solution of the Bargmann system* $(8.2) + (8.16)$ *into the solution of the stationary cKdV equation*

$$Y_N + \hat{c}_{N1} Y_{N-1} + \cdots + \hat{c}_{NN} Y_0 = 0. \tag{8.18}$$

Lemma 8.6. *Let* $(u, v)^T = f_c(p, q)$ *and* t_λ, τ_λ *be the variables of the* F_λ*- and* H_λ*-flow, respectively. Then*

(a)
$$\frac{d}{dt_\lambda}\begin{pmatrix} u \\ v \end{pmatrix} = d f_c(I\nabla F_\lambda) = -2\hat{J}\hat{G}_\lambda. \tag{8.19}$$

(b)
$$\frac{d}{d\tau_\lambda}\begin{pmatrix} u \\ v \end{pmatrix} = d f_c(I\nabla H_\lambda) = \lambda^{-2}\hat{J}\hat{g}_\lambda. \tag{8.20}$$

(c)
$$d f_c(I\nabla H_\lambda) = Y_k. \tag{8.21}$$

Theorem 8.7. *(a) Let* $(p, q)^T$ *be the compatible solution of the* H_0*-flow* (x) *and the* H_k*-flow* (τ_k). *Then* $(u, v)^T = f_c(p, q)$ *satisfies the kth cKdV equation*

$$\frac{d}{d\tau_k}\begin{pmatrix} u \\ v \end{pmatrix} = Y_k.$$

(b) Let $(p(x, y, t), q(x, y, t))^T$ *be the compatible solution of the* H_0*-,* H_1*- and* H_2*-flow (with variables* x, y *and* t*, respectively). Then* $w(x, y, t) = 2v = 2\langle p, q \rangle$ *solves the KP equation (8.11).*

Proof. (a) is a corollary of lemma 8.6, while (b) is obtained by taking into account proposition 8.2. □

Theorem 8.8. *The KP equation (8.11) has the solution*

$$w(x, y, t) = 2\partial_x^2 \ln\theta(x\Omega_0 + y\Omega_1 + t\Omega_2 + D_1) + w_0 \tag{8.22}$$

where $D_1 = \phi_0 + \mathcal{K} + \eta_1$, $w_0 = \sum \alpha_j^2 - I_2(\Gamma) + c$.

Proof. Theorem 8.7 yields the decomposition diagram (1.4) of the KP equation, whose special solution is expressed very simply by the Abel–Jacobi coordinate: $\phi = \phi_0 + x\Omega_0 + y\Omega_1 + t\Omega_2$. What we need to do is to invert it into the usual coordinate.

By making use of (8.2), we have

$$u_x + u^2 + 2v = \langle Aq, q\rangle$$

from $u = \langle q, q\rangle$. On the other hand, they can be expressed through the elliptic coordinates by (5.5) and (5.6):

$$\begin{aligned} u &= \sum_{j=1}^{g}(\alpha_j - \nu_j) \\ u_x + u^2 + 2v &= \tfrac{1}{2}\sum_{j=1}^{g}\left(\alpha_j^2 - \nu_j^2\right) + \tfrac{1}{2}u^2. \end{aligned} \tag{8.23}$$

Since $\{\nu_j\}$ are the zeros of $\theta(\mathcal{A}(P(\lambda)) - \phi - \mathcal{K})$ by Riemann's theorem (see section 7), an ordinary treatment gives [1, 21, 22]

$$\sum_{j=1}^{g}\nu_j = I_1(\Gamma) + \Omega_0^j\partial_j \ln\frac{\theta_1}{\theta_2}$$

$$\sum_{j=1}^{g}\nu_j^2 = I_2(\Gamma) + \Omega_1^j\partial_j \ln\frac{\theta_1}{\theta_2} - \Omega_0^j\Omega_0^k\partial_{jk}^2 \ln\theta_1\theta_2$$

with

$$\theta_s = \theta(x\Omega_0 + y\Omega_1 + t\Omega_2 + D_s) \qquad D_s = \phi_0 + \mathcal{K} + \eta_s$$

where $I_k(\Gamma)$ is given by (7.6), $\partial_j = \partial/\partial\zeta_j$, etc, and the Einstein summation convention is used. Hence

$$\begin{aligned} \sum_{j=1}^{g}\nu_j &= I_1(\Gamma) + \partial_x \ln\frac{\theta_1}{\theta_2} \\ \sum_{j=1}^{g}\nu_j^2 &= I_2(\Gamma) + \partial_y \ln\frac{\theta_1}{\theta_2} - \partial_x^2 \ln\theta_1\theta_2. \end{aligned} \tag{8.24}$$

By making use of the equation for Y_1-flow (8.9) + (8.10)

$$-\partial_{xy}^2 \ln\frac{\theta_1}{\theta_2} = u_y = (u_x + u^2 + 2v)_x$$

we have

$$-\partial_y \ln\frac{\theta_1}{\theta_2} = u_x + u^2 + 2v + c.$$

Put this into (8.24), then substitute (8.24) into (8.23). Finally, we have

$$2v = 2\partial_x^2 \ln\theta_1 + \sum\alpha_j^2 - I_2(\Gamma) + c.$$

This completes the proof. □

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